The Aichelburg–Sexl Boost of an Isolated Source in General Relativity

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Abstract

A study of the Aichelburg–Sexl boost of the Schwarzschild field is described in which the emphasis is placed on the field (curvature tensor) with the metric playing a secondary role. This is motivated by a description of the Coulomb field of a charged particle viewed by an observer whose speed relative to the charge approaches the speed of light. Our approach is exemplified by carrying out an Aichelburg–Sexl type boost on the Weyl vacuum gravitational field due to an isolated axially symmetric source. Detailed calculations of the boosts transverse and parallel to the symmetry axis are given and the results, which differ significantly, are discussed.

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1 Introduction

It was pointed out many years ago by Bergmann [1] that to a fast–moving observer travelling rectilinearly, the Coulomb field of a point charge e, with a time–like geodesic world–line in Minkowskian space–time, resembles the electromagnetic field of a plane electromagnetic wave with a sharpely peaked profile, the closer the speed v of the observer relative to the charge approaches the speed of light. In fact in the limit $v \to 1$ (we shall use units in which the speed of light $c = 1$) the field of the charge seen by the observer is that of a plane impulsive electromagnetic wave [2], [3], i.e. a plane electromagnetic wave having a Dirac delta function profile. The particle origin of this wave is reflected in the fact that the wave front contains a singular point. In other words the history of the wave front in Minkowskian space–time is a null hyperplane generated by parallel null geodesics and on one of these null geodesics the field amplitude is singular. The gravitational analogue of this result is contained in an influential paper by Aichelburg and Sexl [4]. They have shown that to an observer moving rectilinearly relative to a sphere of mass m (the source of the Schwarzschild space–time) with speed v the space–time in the limit $v \to 1$ is a model of a plane impulsive gravitational wave. As in the electromagnetic case the history of this plane gravitational wave contains a null geodesic on which the field amplitude is singular. Many properties of this important result have been elucidated [5] – [8] and it is central to the description in general relativity of the high–speed collision of black–holes [9].

The present paper is a study of the Aichelburg–Sexl result, emphasising the role of the curvature tensor. In all examples considered the Aichelburg–Sexl boost results in a curvature tensor concentrated on a null hyperplane in Minkowskian space–time. The singularity in the curvature is in the form of a delta function which is singular on the null hyperplane. This is consistent with the two halves of Minkowskian space-time (one half is the region to the future of the null hyperplane and the other half is the region to the past of it) being re–attached on the null hyperplane with a matching which preserves the intrinsic metric of the hyperplane. The re–attachment involves a knowledge of a single function defined on the hyperplane. When this function is known the coefficients of the delta function in the curvature are calculated from it. The relevant formulas have been derived in [10] and are reproduced in Appendix A here. Thus for the examples given in this paper the relevant matching is explicitly calculated. It is then possible to express the metric tensor of the re–attached space–time in a coordinate system in which this metric tensor is continuous across the null hyperplane [10]. Having established our point of view we then consider what we call the “Aichelburg–Sexl

boost" of the Weyl, static, axially symmetric, asymptotically flat vacuum gravitational fields. Our results of course depend upon whether the observer is moving parallel to the symmetry axis or not. Boosting parallel to the symmetry axis reproduces the Aichelburg–Sexl space–time in this case while boosting transverse to the symmetry axis produces a plane impulsive gravitational wave which has an amplitude which is singular on one of the null geodesic generators of the history of the plane wave front. This singularity however is more severe than in the monopole case and has a typical multipole character.

As this paper represents a fresh look at the Aichelburg–Sexl boost of the Schwarzschild space–time we want to make it as self–contained as possible. To this end and to make clear our point of view the paper is organised as follows: the boost of the Coulomb field described at the beginning of this introduction is given in detail in section 2 and then in section 3 we rederive the Aichelburg–Sexl result from our point of view. This prepares the reader for our description in section 4 of the boosted Weyl asymptotically flat fields. The paper ends with a discussion in section 5. Some fairly extensive calculations accompany our work and these are summarised in the appendices.

2 Coulomb Field

We begin with the line-element of Minkowskian space–time in rectangular Cartesian coordinates and time $\{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$,

$$ds^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{t}^2 \ . \tag{2.1}$$

We are using units in which the speed of light $c = 1$. Let the $\bar{t}$–axis ($\bar{x} = \bar{y} = \bar{z} = 0$) be the time–like geodesic world–line of a charge e. The Coulomb potential of this charge at $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ is given by the 1–form

$$A = -\frac{e}{\bar{r}}\, d\bar{t} \ , \qquad \bar{r} = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2} \ . \tag{2.2}$$

The corresponding electric field is given by the 2–form

$$F = dA = \frac{e}{\bar{r}^3}\, (\bar{x}\, d\bar{x} \wedge d\bar{t} + \bar{y}\, d\bar{y} \wedge d\bar{t} + \bar{z}\, d\bar{z} \wedge d\bar{t}) \ . \tag{2.3}$$

The field measured by an observer moving in the $-\bar{x}$ direction with speed v relative to the charge is obtained by making in (2.3) the Lorentz transformation

$$\bar{x} = \gamma\, (x - v\, t) \ , \qquad \bar{y} = y \ , \qquad \bar{z} = z \ , \qquad \bar{t} = \gamma\, (t - v\, x) \ , \tag{2.4}$$

with $\gamma = (1 - v^2)^{-1/2}$. Thus we can write

$$F = \frac{e\,\gamma^{-2}}{R^3}\,\{(x - v\,t)\,dx \wedge dt + (y\,dy + z\,dz) \wedge (dt - v\,dx)\}\ , \tag{2.5}$$

with

$$R = \left\{(x - v\,t)^2 + \gamma^{-2}(y^2 + z^2)\right\}^{1/2}\ . \tag{2.6}$$

We want the limit of (2.5) as $v \rightarrow 1$. To this end we make the simple observation:

$$\frac{\gamma^{-2}}{R^3} = \frac{1}{y^2 + z^2}\,\frac{\partial}{\partial x}\left(\frac{x - v\,t}{R}\right)\ , \tag{2.7}$$

and so

$$\lim_{v \longrightarrow 1} \frac{\gamma^{-2}}{R^3} = (y^2 + z^2)^{-1}\frac{\partial}{\partial x}\left(\frac{x - t}{|x - t|}\right)\ . \tag{2.8}$$

Denoting by $\vartheta(u)$ the Heaviside step function which we take to be unity if $u > 0$ and zero if $u < 0$ we can write

$$\frac{x - t}{|x - t|} = 2\,\vartheta(x - t) - 1\ , \tag{2.9}$$

and so (2.8) can be written

$$\lim_{v \longrightarrow 1} \frac{\gamma^{-2}}{R^3} = \frac{2\,\delta(x - t)}{y^2 + z^2}\ , \tag{2.10}$$

with $\delta(x - t)$ the Dirac delta function singular on the null hyperplane $x = t$. Remembering that $(x - t)\,\delta(x - t) = 0$ we see now from (2.5) that

$$\lim_{v \longrightarrow 1} F = \frac{2\,e\,\delta(x - t)}{y^2 + z^2}\,(y\,dy + z\,dz) \wedge (dt - dx) = F_0\ (\text{say})\ . \tag{2.11}$$

Clearly we can write

$$F_0 = dA_0\ , \qquad \text{with} \qquad A_0 = e\,\delta(x - t)\,\log(y^2 + z^2)\,(dt - dx)\ , \tag{2.12}$$

and so F_0 is a solution of the source–free Maxwell equations which is singular on $x = t$ and also on the null geodesic generator of $x = t$ labelled by $y = z = 0$. F_0 describes an impulsive electromagnetic wave (the 2–form F_0 is Type N in the Petrov classification with degenerate principal null direction given by the 1–form $dx - dt$) with profile $\delta(x - t)$.

Substituting the Lorentz transformation (2.4) into the potential 1–form (2.2) yields

$$A = -\frac{e}{R}\,(dt - v\,dx)\ , \tag{2.13}$$

with R given by (2.6). From this we arrive at

$$\lim_{v\longrightarrow 1} A = -e\,\frac{(dt-dx)}{|t-x|}\ , \tag{2.14}$$

which, of course, only makes sense if $x \neq t$. The limit (2.14), being a pure gauge term for $x > t$ and for $x < t$, is consistent with (2.11). To see what happens *on* $x = t$ in the limit $v \to 1$ we first modify the Coulomb potential (2.2) by the addition of a gauge term to read (this gauge term is suggested by a clever analogous coordinate transformation in the gravitational case [4])

$$A = -\frac{e}{\bar{r}}\,d\bar{t} - \frac{e\,d\bar{x}}{\sqrt{\bar{x}^2+1}}\ . \tag{2.15}$$

This potential 1–form obviously leads to the same electric field (2.3) as that derived from (2.7). The Lorentz transformation (2.4) applied to (2.15) yields

$$A = -\frac{e}{R}\,(dt - v\,dx) - \frac{e\,(dx - v\,dt)}{\{(x-v\,t)^2+\gamma^{-2}\}^{1/2}}\ . \tag{2.16}$$

Now

$$dx - v\,dt = -dt + v\,dx + (1-v)(dx+dt)\ , \tag{2.17}$$

and so for v near 1 we can write (2.16) as

$$A = -e\left[\frac{1}{R} - \frac{1}{\{(x-v\,t)^2+\gamma^{-2}\}^{1/2}}\right](dt - v\,dx) + O\left((1-v)\right)\ . \tag{2.18}$$

This can further be written

$$A = -e\,\frac{\partial}{\partial x}\left[\log\left(\frac{x-v\,t+R}{x-v\,t+\sqrt{(x-v\,t)^2+\gamma^{-2}}}\right)\right](dt - v\,dx) + O\left((1-v)\right)\ . \tag{2.19}$$

The logarithm term here appeared first in [4] and it is particularly useful when one observes that [4]

$$\lim_{v\longrightarrow 1}\log\left(\frac{x-v\,t+R}{x-v\,t+\sqrt{(x-v\,t)^2+\gamma^{-2}}}\right) = \left(1-\vartheta(x-t)\right)\,\log(y^2+z^2)\ . \tag{2.20}$$

Hence (2.19) gives

$$\lim_{v\longrightarrow 1} A = e\,\frac{\partial}{\partial x}\Big(\vartheta(x-t)\,\log(y^2+z^2)\Big)\,(dt-dx) = e\,\delta(x-t)\,\log(y^2+z^2)\,(dt-dx)\ , \tag{2.21}$$

and so we have recovered A_0 in (2.12). A comprehensive study of light–like boosts of electromagnetic multipole fields (which has little overlap in approach with our discussion in section 4 below of gravitational multipole fields) has been carried out by Robinson and Rózga [11].

We regard the limit of the field (2.11) to be the important result from which the physical interpretation (that the boosted Coulomb field is the field of an impulsive electromagnetic wave in the limit $v \to 1$) follows. The limit of the potential is in this sense of secondary importance. Thus in the gravitational case we shall place an emphasis on boosting the space–time curvature while the metric will play a secondary role.

3 Aichelburg–Sexl Boost

As in [4] we take the Schwarzschild line–element in isotropic coordinates as starting point:

$$ds^2 = (1+A)^4\,(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) - \frac{(1-A)^2}{(1+A)^2}\,d\bar{t}^2\ , \tag{3.1}$$

with

$$A = \frac{m}{2\,\bar{r}}\ , \qquad \bar{r} = \{\bar{x}^2 + \bar{y}^2 + \bar{z}^2\}^{1/2}\ . \tag{3.2}$$

The constant m is the mass of the source. Any other asymptotically flat form of the line–element in which the coordinates are asymptotically rectangular Cartesians and time will do. For example one could start with the Kerr–Schild form of (3.1) in coordinates $\{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$ in terms of which the flat background line–element is given by (2.1). Since we wish to emphasise the Riemann curvature tensor we require the curvature tensor components $\bar{R}_{ijkl}$ of the space–time with line–element (3.1) in coordinates $\{\bar{x}^i\} = \{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$. The non–identically vanishing components are

$$\begin{aligned}
\bar{R}_{1212} &= -\frac{m(\bar{r}^2 - 3\,\bar{z}^2)}{\bar{r}^5}\,(1+A)^2\ , \qquad \bar{R}_{1313} = -\frac{m(\bar{r}^2 - 3\,\bar{y}^2)}{\bar{r}^5}\,(1+A)^2\ ,\\
\bar{R}_{2323} &= -\frac{m(\bar{r}^2 - 3\,\bar{x}^2)}{\bar{r}^5}\,(1+A)^2\ , \qquad \bar{R}_{1213} = -\frac{3\,m\,\bar{y}\,\bar{z}}{\bar{r}^5}\,(1+A)^2\ ,\\
\bar{R}_{1223} &= \frac{3\,m\,\bar{x}\,\bar{z}}{\bar{r}^5}\,(1+A)^2\ , \qquad \bar{R}_{1323} = -\frac{3\,m\,\bar{x}\,\bar{y}}{\bar{r}^5}\,(1+A)^2\ ,\\
\bar{R}_{1414} &= \frac{m\,(\bar{r}^2 - 3\,\bar{x}^2)}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad \bar{R}_{2424} = \frac{m\,(\bar{r}^2 - 3\,\bar{y}^2)}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ ,\\
\bar{R}_{3434} &= \frac{m\,(\bar{r}^2 - 3\,\bar{z}^2)}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad \bar{R}_{1424} = -\frac{3\,m\,\bar{x}\,\bar{y}}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ ,
\end{aligned}$$

$$\bar{R}_{1434} = -\frac{3\,m\,\bar{x}\,\bar{z}}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad \bar{R}_{2434} = -\frac{3\,m\,\bar{y}\,\bar{z}}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \tag{3.3}$$

Now make the Lorentz transformation (2.4). If $\{x^i\} = \{x, y, z, t\}$ then the non–identically vanishing components R_{ijkl} of the Riemann tensor in the unbarred coordinates are related to (3.3) via

$$\begin{aligned}
R_{1212} &= \gamma^2\left(\bar{R}_{1212} + v^2\bar{R}_{2424}\right)\ , \qquad R_{1313} = \gamma^2\left(\bar{R}_{1313} + v^2\bar{R}_{3434}\right)\ ,\\
R_{2124} &= -\gamma^2 v\left(\bar{R}_{2121} + \bar{R}_{2424}\right)\ , \qquad R_{3134} = -\gamma^2 v\left(\bar{R}_{3131} + \bar{R}_{3434}\right)\ ,\\
R_{3124} &= -\gamma^2 v\left(\bar{R}_{3121} + \bar{R}_{3424}\right)\ , \qquad R_{1234} = \gamma^2 v\left(\bar{R}_{3121} + \bar{R}_{3424}\right)\ ,\\
R_{2434} &= \gamma^2\left(\bar{R}_{2434} + v^2\bar{R}_{1213}\right)\ , \qquad R_{1213} = \gamma^2\left(\bar{R}_{1213} + v^2\bar{R}_{2434}\right)\ ,\\
R_{1223} &= \gamma\,\bar{R}_{1223}\ , \qquad R_{1323} = \gamma\bar{R}_{1323}\ , \qquad R_{1414} = \bar{R}_{1414}\ ,\\
R_{2424} &= \gamma^2\left(\bar{R}_{2424} + v^2\bar{R}_{1212}\right)\ , \qquad R_{3434} = \gamma^2\left(\bar{R}_{3434} + v^2\bar{R}_{1313}\right)\ ,\\
R_{1424} &= \gamma\,\bar{R}_{1424}\ , \qquad R_{1434} = \gamma\,\bar{R}_{1434}\ , \qquad R_{2323} = \bar{R}_{2323}\ .
\end{aligned} \tag{3.4}$$

Of course when (3.3) is now substituted into (3.4) the barred coordinates in (3.3) are expressed in terms of the unbarred coordinates using (2.4). We now make the Aichelburg–Sexl boost of (3.4) by taking the limit $v \rightarrow 1$. In this limit the rest mass $m \rightarrow 0$ and $\gamma \rightarrow \infty$ in such a way that $m\,\gamma = p$ (say) remains finite and plays the role of the energy of the source. The calculation of the limit is straightforward and makes use of

$$\lim_{v\longrightarrow 1}\frac{\gamma^{-4}}{R^5} = \frac{4}{3}\,\frac{\delta(x-t)}{(y^2+z^2)^2}\ , \tag{3.5}$$

which is obtained from (2.10) by differentiating with respect to y (or z) assuming y, z are non–zero. If we write

$$\tilde{R}_{ijkl} = \lim_{v\longrightarrow 1} R_{ijkl}, \tag{3.6}$$

we find that $\tilde{R}_{ijkl} \equiv 0$ except for

$$\begin{aligned}
\tilde{R}_{1212} &= \tilde{R}_{2424} = -\tilde{R}_{1313} = -\tilde{R}_{3434} =\\
\tilde{R}_{3134} &= -\tilde{R}_{2124} = -\frac{4\,p\,(y^2-z^2)\,\delta(x-t)}{(y^2+z^2)^2}\ ,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\tilde{R}_{1213} &= \tilde{R}_{2434} = -\tilde{R}_{3124} = -\tilde{R}_{2134}\\
&= -\frac{8\,p\,yz\,\delta(x-t)}{(y^2+z^2)^2}\ ,
\end{aligned} \tag{3.8}$$

We note that since $R_{jk}=0$ we have $\tilde{R}_{jk}=\lim_{v\to 1}R_{jk}\equiv 0$. Substitution of the Lorentz transformation (2.4) into the line–element (3.1) and then taking the limit $v\to 1$ gives, in agreement with [4],

$$\lim_{v\longrightarrow 1} ds^2 = dx^2+dy^2+dz^2-dt^2+\frac{4\,p}{|x-t|}\,(dt-dx)^2\ . \tag{3.9}$$

This holds for $x\neq t$ and is the analogue of (2.14). For $x>t$ and for $x<t$ this is the line–element of Minkowskian space– time (which of course is consistent with (3.7) and (3.8)). For $x>t$ we can write (3.9) in the form

$$ds_+^2 = dy_+^2+dz_+^2+2\,du\,dv_+\ , \tag{3.10}$$

with

$$y_+=y\ ,\qquad z_+=z\ ,\qquad u=x{-}t\ ,\qquad v_+=\frac{1}{2}(x{+}t){+}2p\,\log(x{-}t)\ . \tag{3.11}$$

For $x<t$ we can write (3.9) in the form

$$ds_-^2 = dy_-^2+dz_-^2+2\,du\,dv_-\ , \tag{3.12}$$

with

$$y_-=y\ ,\qquad z_-=z\ ,\qquad u=x{-}t\ ,\qquad v_-=\frac{1}{2}(x{+}t){-}2p\,\log(t{-}x)\ . \tag{3.13}$$

¿From (3.10) and (3.12) we see that $u=x-t=0$ is a null hyperplane in Minkowskian space–time. The line–elements (3.10) and (3.12) are consistent with having a delta function in the Riemann curvature tensor which is singular on $x=t$ provided the two halves of Minkowskian space–time, $x>t$ and $x<t$, are attached on $x=t$ with [10]

$$y_+=y_-\ ,\qquad z_+=z_-\ ,\qquad v_+=F(v_-,y_-,z_-)\ , \tag{3.14}$$

for some function F defined on $x=t$ for which $\partial F/\partial v_-\neq 0$. In the general case , for any such F, this will make $x=t$ a model of the most general plane–fronted light–like signal propagating through flat space–time with a delta function in the curvature tensor singular on $x=t$ and with, in general, a delta function in the Ricci tensor aswell (so that the signal could be a plane–fronted impulsive gravitational wave accompanied by a plane–fronted light–like shell of null matter). The coefficients of the delta function in the Riemann tensor and the Einstein tensor are constructed from derivatives of the function F in (3.14). The explicit formulas from [10] are listed in Appendix A. The line–element of the re–attached halves of Minkowskian space–time can be

presented, once F is known, in a coordinate system in which the metric tensor is continuous across the singular hyperplane. This is given in [10]. Thus the consistency of (3.14) with the calculated curvature tensor components (3.7) and (3.8) implies that in the present case (see Appendix A)

$$v_+ = F = v_- + 2p\,\log(y^2+z^2)\ , \tag{3.15}$$

and that the signal in this case with history $x=t$ is an impulsive gravitational wave unaccompanied by a light–like shell (because $\tilde{R}_{jk}\equiv 0$). Moreover the function F in (3.15) is unique up to a (trivial) change of affine parameter v_+ along the generators of $x=t$. Finally we note from (3.7) and (3.8) that the amplitudes of the field components (the coefficients of the delta function) are singular on the generator $y=z=0$ of the null hyperplane $x=t$. This is the remnant of the particle origin of the wave described by (3.7) and (3.8).

4 Boosting an Isolated Source

To illustrate the usefulness of our approach in which the Riemann curvature tensor plays a central role we consider the Weyl, static, axially symmetric, asymptotically flat solutions of Einstein's vacuum field equations. The line–element is given by

$$ds^2 = \mathrm{e}^{2\sigma-2\psi}(d\bar{r}^2+\bar{r}^2 d\bar{\theta}^2) + \mathrm{e}^{-2\psi}\bar{r}^2\sin^2\bar{\theta}\,d\bar{\phi}^2 - \mathrm{e}^{2\psi}\,d\bar{t}^2\ , \tag{4.1}$$

with

$$\psi = \sum_{l=0}^{\infty}\frac{A_l}{\bar{r}^{l+1}}\,P_l(\cos\bar{\theta})\ , \tag{4.2}$$

$$\sigma = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{A_l\,A_m\,(l+1)\,(m+1)}{(l+m+2)\,\bar{r}^{l+m+2}}\,\left(P_{l+1}\,P_{m+1}-P_l\,P_m\right)\ . \tag{4.3}$$

Here $P_l(\cos\bar{\theta})$ is the Legendre polynomial of degree l in the variable $\cos\bar{\theta}$ and the A_l are constants related to the multipole moments of the source. For example [13] $A_0=-m\ , A_1=-D\ , A_2=-Q-\frac{1}{3}m^3$ where m is the non–zero rest mass of the source, D is its dipole moment and Q is its quadrupole moment. We can introduce coordinates $\{\bar{x},\bar{y},\bar{z}\}$ so that $\{\bar{x},\bar{y},\bar{z},\bar{t}\}$ are asymptotically rectangular Cartesians and time simply by putting

$$\bar{x}=\bar{r}\,\sin\bar{\theta}\,\sin\bar{\phi}\ ,\qquad \bar{y}=\bar{r}\,\sin\bar{\theta}\,\cos\bar{\phi}\ ,\qquad \bar{z}=\bar{r}\,\cos\bar{\theta}\ . \tag{4.4}$$

Then we can write (4.1) in the form

$$ds^2 = g_{AB}d\bar{x}^A\,d\bar{x}^B + \mathrm{e}^{2\sigma-2\psi}d\bar{z}^2 - \mathrm{e}^{2\psi}d\bar{t}^2\ , \tag{4.5}$$

with $\{\bar{x}^A\} = \{\bar{x}, \bar{y}\}$ and

$$(g_{AB}) = \frac{\mathrm{e}^{-2\psi}}{(\bar{x}^2 + \bar{y}^2)} \begin{pmatrix} \mathrm{e}^{2\sigma}\bar{x}^2 + \bar{y}^2 & (\mathrm{e}^{2\sigma} - 1)\,\bar{x}\,\bar{y} \\ (\mathrm{e}^{2\sigma} - 1)\,\bar{x}\,\bar{y} & \mathrm{e}^{2\sigma}\bar{y}^2 + \bar{x}^2 \end{pmatrix} ,$$

and capital letters take values $1, 2$. The Legendre polynomials are now functions of $\bar{z}/\bar{r}$ with $\bar{r} = \{\bar{x}^2 + \bar{y}^2 + \bar{z}^2\}^{1/2}$.

We will consider a Lorentz boost in the $-\bar{x}$ direction given by (2.4) and also a Lorentz boost in the $-\bar{z}$ direction (along the symmetry axis) given by

$$\bar{x} = x \ , \qquad \bar{y} = y \ , \qquad \bar{z} = \gamma(z - v\,t) \ , \qquad \bar{t} = \gamma(t - v\,z) \ . \tag{4.6}$$

We will then follow this by taking the limit $v \to 1$ (which we are calling the Aichelburg–Sexl boost). In the monopole case considered by Aichelburg and Sexl the rest mass m of the source was taken to tend to zero in this limit in such a way that the energy $p = m\,\gamma$ remained finite. Now the behavior of all of the constants A_l ($l = 0, 1, 2, \ldots$), related to the multipole moments of the source, in the limit $v \to 1$, has to be considered. A simple guide is the special case of the Curzon [14] solution which is the subcase of (4.1)–(4.3) corresponding to two point masses having rest–masses m_1 , m_2 located on the $\bar{z}$–axis at $\bar{z} = \pm a$ respectively and held in position with a strut. In this case

$$A_l = -m_1\,a^l - m_2\,(-a)^l \ , \qquad l = 0, 1, 2, \ldots \ . \tag{4.7}$$

In any boost we will assume as before that m_1, m_2 tend to zero as $v \to 1$ like γ^{-1}, i.e.

$$m_1 = \gamma^{-1}\hat{p}_1 \ , \qquad m_2 = \gamma^{-1}\hat{p}_2 \ , \tag{4.8}$$

with $\hat{p}_1$, $\hat{p}_2$ independent of v. Thus for a Lorentz boost in the $-\bar{x}$ direction,

$$A_l = \gamma^{-1}p_l \ , \qquad l = 0, 1, 2, \ldots \ , \tag{4.9}$$

with

$$p_l = -\hat{p}_1\,a^l - \hat{p}_2\,(-a)^l \ . \tag{4.10}$$

On the other hand for a Lorentz boost in the $-\bar{z}$ direction we assume that a tends to zero as $v \to 1$ like γ^{-1}, i.e.

$$a = \gamma^{-1}\hat{a} \ , \tag{4.11}$$

with $\hat{a}$ independent of v. Hence in this case

$$A_l = \gamma^{-l-1}\,p_l \ , \qquad l = 0, 1, 2, \ldots \ , \tag{4.12}$$

with now

$$p_l = -\hat{p}_1\,\hat{a}^l - \hat{p}_2(-\hat{a})^l \ . \tag{4.13}$$

The essential difference between (4.9) and (4.12) is due to the fact that (4.12) incorporates the Lorentz contraction (4.11). We shall in general consider only a class of vacuum gravitational fields described by (4.1)–(4.3) for which A_l scales with γ in the form of (4.9) for a Lorentz boost in the (transverse) $-\bar{x}$ direction and in the form (4.12) for a boost parallel to the axis of symmetry (the $-\bar{z}$ direction). Since we shall be considering later the limit $v \rightarrow 1$ and thus $\gamma^{-1} \rightarrow 0$ we can slightly weaken our assumptions by taking (4.9) and (4.12) to be the leading terms in A_l in the two cases for small γ^{-1} (thus allowing additional terms which go to zero faster than γ^{-1} in the limit $v \rightarrow 1$) without affecting the outcome of our calculations.

We begin by applying the Lorentz transformation (2.4) in the $-\bar{x}$ direction to the curvature tensor calculated from the line–element (4.5). The components $\bar{R}_{ijkl}$ of this curvature tensor, in the coordinates $\{\bar{x}\ ,\bar{y}\ ,\bar{z}\ ,\bar{t}\ \}$ are given in Appendix B. These are then transformed to R_{ijkl} in the coordinates $\{x\ ,y\ ,z\ ,t\}$ introduced in (2.4). The components R_{ijkl} are given in (B.5). We then finally calculate the limit of these components as $v \rightarrow 1$, which we denote by

$$\tilde{R}_{ijkl} = \lim_{v \longrightarrow 1} R_{ijkl}\ . \tag{4.14}$$

We note that since the space–time described by (4.1)–(4.3) is a vacuum space–time, the Ricci tensor R_{jk} vanishes and so $\tilde{R}_{jk} = \lim_{v\rightarrow 1} R_{jk} = 0$. In carrying out this programme of calculations everything hinges on the functions $\psi\ ,\sigma$ and their first and second derivatives with respect to $\bar{x}\ ,\bar{y}\ ,\bar{z}$ which must be evaluated in the coordinates $\{x\ ,y\ ,z\ ,t\}$ for substitution into (B.5). With $A_l = \gamma^{-1}p_l\ ,\ l = 0, 1, 2, \ldots$, as in (4.9), and

$$\bar{r} = \gamma\, R\ , \qquad R = \sqrt{(x - v\,t)^2 + \gamma^{-2}(y^2 + z^2)}\ , \tag{4.15}$$

and starting with

$$\psi = \sum_{l=0}^{\infty} \frac{A_l}{\bar{r}^{l+1}}\, P_l\left(\frac{\bar{z}}{\bar{r}}\right)\ , \tag{4.16}$$

we can write

$$\psi = \sum_{l=0}^{\infty} p_l\, \frac{\gamma^{-l-2}}{R^{l+1}}\, P_l(w)\ , \tag{4.17}$$

with

$$w = \gamma^{-1}\frac{z}{R}\ . \tag{4.18}$$

It is clear from (4.17) that as $v \rightarrow 1$, $\psi \rightarrow 0$ like γ^{-2} for $x \neq t$ and $\psi \rightarrow 0$ like γ^{-1} for $x = t$. Starting with (4.16) we obtain

$$\frac{\partial\psi}{\partial\bar{x}} \ \ = \ \ -\sum_{l=0}^{\infty} p_l\,(x - v\,t)\,\frac{\gamma^{-l-3}}{R^{l+3}}\,P'_{l+1}\ , \tag{4.19}$$

$$
\frac{\partial\psi}{\partial\bar{y}} = -\sum_{l=0}^{\infty} p_l\, y\, \frac{\gamma^{-l-4}}{R^{l+3}}\, P'_{l+1}\ , \tag{4.20}
$$

$$
\frac{\partial\psi}{\partial\bar{z}} = -\sum_{l=0}^{\infty} p_l\, (l+1)\, \frac{\gamma^{-l-3}}{R^{l+2}}\, P_{l+1}\ , \tag{4.21}
$$

where the argument of the Legendre polynomials is w given in (4.18) and the prime denotes differentiation with respect to w. We have simplified (4.19) and (4.20) using

$$
w\, P'_l + (l+1)\, P_l = P'_{l+1}\ , \tag{4.22}
$$

for $l = 0, 1, 2, \ldots$, and (4.21) has been simplified using

$$
(w^2 - 1)\, P'_l = (l+1)\ (P_{l+1} - w\, P_l)\ \ , \tag{4.23}
$$

for $l = 0, 1, 2, \ldots$. We need the second derivatives of ψ and making use of the relations (4.22) and (4.23) satisfied by the Legendre polynomials we arrive at

$$
\frac{\partial^2\psi}{\partial\bar{x}^2} = -\sum_{l=0}^{\infty} p_l\, \frac{\partial}{\partial x}\left(\frac{(x - v\, t)\, \gamma^{-l-4} P'_{l+1}}{R^{l+3}}\right)\ , \tag{4.24}
$$

$$
\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}} = -\sum_{l=0}^{\infty} p_l\, (x - v\, t)\, \frac{\partial}{\partial y}\left(\frac{\gamma^{-l-3} P'_{l+1}}{R^{l+3}}\right)\ , \tag{4.25}
$$

$$
\frac{\partial^2\psi}{\partial\bar{y}^2} = -\sum_{l=0}^{\infty} p_l\, \frac{\partial}{\partial y}\left(\frac{y\, \gamma^{-l-4} P'_{l+1}}{R^{l+3}}\right)\ , \tag{4.26}
$$

$$
\frac{\partial^2\psi}{\partial\bar{z}^2} = \sum_{l=0}^{\infty} p_l\, (l+1)\, (l+2)\, \frac{\gamma^{-l-4} P_{l+2}}{R^{l+3}}\ , \tag{4.27}
$$

$$
\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{z}} = \sum_{l=0}^{\infty} p_l\, (l+1)\, (x - v\, t)\, \frac{\gamma^{-l-4} P'_{l+2}}{R^{l+4}}\ , \tag{4.28}
$$

$$
\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}} = \sum_{l=0}^{\infty} p_l\, (l+1)\, y\, \frac{\gamma^{-l-5} P'_{l+2}}{R^{l+4}}\ . \tag{4.29}
$$

To evaluate the limit $v \to 1$ of quantities involving the derivatives (4.19)–(4.21) and (4.24)–(4.29) we make use of the following:

$$
\lim_{v\to 1} \frac{\gamma^{-l} P_l(w)}{R^{l+1}} = \frac{(-1)^{l+1}}{l!}\, g^{(l)}\, \delta(x - t)\ , \tag{4.30}
$$

for $l = 1, 2, 3, \ldots$, with $g = \log(y^2 + z^2)$ and $g^{(l)} = \partial^l g/\partial z^l$. This is established by induction on l. It is true for $l = 1$ because

$$
\frac{\gamma^{-1} P_1(w)}{R^2} = \frac{\gamma^{-1} w}{R^2} = \frac{\gamma^{-2} z}{R^3}\ , \tag{4.31}
$$

and so by (2.10)

$$\lim_{v\to1}\frac{\gamma^{-1}P_1(w)}{R^2}=\frac{2\,z}{y^2+z^2}\,\delta(x-t)=\frac{\partial g}{\partial z}\,\delta(x-t)\ . \tag{4.32}$$

Assume (4.30) holds for l and differentiate it with respect to z to obtain

$$\lim_{v\to1}\frac{\gamma^{-l-1}}{R^{l+2}}\left\{-(l+1)\,w\,P_l+(1-w^2)\,P_l'\right\}=\frac{(-1)^{l+1}}{l!}g^{(l+1)}\,\delta(x-t)\ . \tag{4.33}$$

Using (4.23) this can be simplified to read

$$\lim_{v\to1}\frac{\gamma^{-l-1}P_{l+1}(w)}{R^{l+2}}=\frac{(-1)^{l+2}}{(l+1)!}g^{(l+1)}\,\delta(x-t)\ , \tag{4.34}$$

and (4.30) holds for $(l+1)$ if it holds for l.

In addition to (4.30) we shall require one more result:

$$\lim_{v\to1}\frac{\gamma^{-l-2}P_{l+1}'(w)}{R^{l+3}}=\frac{2\,(-1)^l}{l!}\,h^{(l)}\,\delta(x-t)\ , \tag{4.35}$$

for $l=0,1,2,\dots$, with $h=(y^2+z^2)^{-1}$ and $h^{(l)}=\partial^l h/\partial z^l$. This can be established by induction on l or more directly by differentiating (4.30) with respect to y to obtain

$$-\lim_{v\to1}\frac{\gamma^{-l-2}y}{R^{l+3}}\ \{(l+1)\,P_l+w\,P_l'\}=\frac{(-1)^{l+1}}{l!}\,\frac{\partial}{\partial y}\left(g^{(l)}\right)\,\delta(x-t)\ , \tag{4.36}$$

for $l=1,2,3,\dots$. If we use (4.22) in the left hand side here and in the right hand side use

$$\frac{\partial}{\partial y}\left(g^{(l)}\right)=\frac{\partial^l}{\partial z^l}\left(\frac{\partial g}{\partial y}\right)=2\,y\,\frac{\partial^l}{\partial z^l}(y^2+z^2)^{-1}=2\,y\,h^{(l)}\ , \tag{4.37}$$

then, assuming $y\neq0$, (4.36) becomes (4.35) for $l=1,2,3,\dots$. That (4.35) also holds for $l=0$ follows from

$$\lim_{v\to1}\frac{\gamma^{-2}P_1'(w)}{R^3}=\lim_{v\to1}\frac{\gamma^{-2}}{R^3}=2\,h\,\delta(x-t)\ , \tag{4.38}$$

with the last equality coming from (2.10).

With the help of (4.30) and (4.35) limits of the derivatives (4.19)–(4.21) and (4.24)–(4.29) can be evaluated which are required for the calculation of $\tilde{R}_{ijkl}$ in (4.14). The reader can easily evaluate the limits needed with the

information given above. We shall mention here, as a guide, some of the required limits. For example using (4.27) and (4.30) we obtain

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{z}^2}=\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}g^{(l+2)}\,\delta(x-t)\ . \tag{4.39}$$

By (4.28) and (4.35),

$$\lim_{v\to 1}\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{z}}=0\ , \tag{4.40}$$

and we have here made use of $(x-t)\,\delta(x-t)=0$. Using (4.29) and (4.35) we find that

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}}=\sum_{l=0}^{\infty}p_l\,\frac{2\,(-1)^{l+1}}{l!}\,y\,h^{(l+1)}\,\delta(x-t)\ . \tag{4.41}$$

We can write

$$2\,y\,h^{(l+1)}=\frac{\partial^{l+1}}{\partial z^{l+1}}\left(\frac{2\,y}{y^2+z^2}\right)=\frac{\partial^{l+1}}{\partial z^{l+1}}\left(\frac{\partial g}{\partial y}\right)=\frac{\partial^2}{\partial z\partial y}\,g^{(l)}\ , \tag{4.42}$$

and thus (4.41) becomes

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}}=\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial z\partial y}\,\delta(x-t)\ . \tag{4.43}$$

Similarly from (4.24), (4.25) using (4.35) and $(x-t)\,\delta(x-t)=0$ we find that

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{x}^2}=0=\lim_{v\to 1}\gamma\,\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}}\ . \tag{4.44}$$

Finally by (4.26) using (4.35),

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}^2}=\sum_{l=0}^{\infty}p_l\,\frac{2\,(-1)^{l+1}}{l!}\,\frac{\partial}{\partial y}\left(y\,h^{(l)}\right)\,\delta(x-t)\ . \tag{4.45}$$

But

$$\frac{\partial}{\partial y}\left(2\,y\,h^{(l)}\right)=\frac{\partial^{l+1}}{\partial y\partial z^{l}}\left(\frac{2\,y}{y^2+z^2}\right)=\frac{\partial^2 g^{(l)}}{\partial y^2}\ , \tag{4.46}$$

and so (4.45) reads

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}^2}=\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial y^2}\,\delta(x-t)\ . \tag{4.47}$$

There remains to be considered the function σ given by (4.3). It is clear from (4.3) that σ is a linear combination with constant coefficients of ψ^2–terms. Hence as $v \to 1$ we have $\sigma \to 0$ like γ^{-4} for $x \neq t$ (compared with γ^{-2} for ψ) and $\sigma \to 0$ like γ^{-2} for $x = t$ (compared with γ^{-1} for ψ). In addition all first and second derivatives of σ *vanish* in the limit $v \to 1$, including the derivatives multiplied by γ or γ^2 (such as the left hand sides of (4.39)–(4.41) and (4.43)–(4.45) with ψ replaced by σ) which are required for the calculation of $\tilde{R}_{ijkl}$ in (4.14). We are now ready to calculate $\tilde{R}_{ijkl}$. We first find that the following components are non–zero:

$$\begin{aligned}
\tilde{R}_{2434} &= \tilde{R}_{2131} = -\tilde{R}_{2134} = -\tilde{R}_{3124} = 2\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{z}\partial\bar{y}}\ , \\
&= 2\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial y\partial z}\,\delta(x-t)\ ,
\end{aligned} \tag{4.48}$$

by (4.43). Next we find

$$\begin{aligned}
\tilde{R}_{2424} &= \tilde{R}_{2121} = -\tilde{R}_{2124} = \lim_{v\to 1}\gamma^2\left(2\,\frac{\partial^2\psi}{\partial\bar{y}^2}+\frac{\partial^2\psi}{\partial\bar{x}^2}\right)\ , \\
&= 2\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial y^2}\,\delta(x-t)\ ,
\end{aligned} \tag{4.49}$$

by (4.44) and (4.47). And finally we obtain

$$\begin{aligned}
\tilde{R}_{3434} &= \tilde{R}_{3131} = -\tilde{R}_{3134} = \lim_{v\to 1}\gamma^2\left(2\,\frac{\partial^2\psi}{\partial\bar{z}^2}+\frac{\partial^2\psi}{\partial\bar{x}^2}\right)\ , \\
&= 2\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial z^2}\,\delta(x-t)\ ,
\end{aligned} \tag{4.50}$$

by (4.39) and (4.44). Noting that $g = \log(y^2+z^2)$ is a harmonic function for $y^2+z^2 \neq 0$ so that

$$\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2} = 0\ , \tag{4.51}$$

the right hand side of (4.50) is minus the right hand side of (4.49). We have now arrived at the analogue of (3.7) and (3.8) in the monopole case and we see that the singularity in (4.48)–(4.50) on $y = z = 0$ is more severe than in the monopole case, involving derivatives of g greater than the second.

Taking the limit $v \to 1$ of the line–element (4.5) following the coordinate transformation (2.4) results in

$$\lim_{v\to 1} ds^2 = dx^2+dy^2+dz^2-dt^2-\frac{4\,p_0}{|x-t|}\,(dt-dx)^2\ , \tag{4.52}$$

for $x \neq t$ (we note that $p_0 = -p$ in (3.9)). We note that only p_0 appears in (4.52) and so (4.52) inherits information only from the monopole structure of the source. The curvature tensor (4.48)–(4.50) is influenced by all the multipole moments of the source and thus contains more information than (4.52). The discussion following (3.9) applies again here. The hyperplane $x = t$ is null and the Riemann tensor $\tilde{R}_{ijkl}$ given by (4.48)–(4.50) is consistent with re–attaching the two halves of Minkowskian space– time $x > t$ and $x < t$ with the matching (3.14) provided the function F in (3.14) is correctly chosen. The formulas for determining F are obtained by equating to zero the components (A.1)– (A.4) of the surface stress–energy tensor (which as before is zero because $\tilde{R}_{jk} = 0$; the signal with history $x = t$ is a pure impulsive gravitational wave) and by equating the expressions for $\tilde{R}_{ijkl}$ in (A.5) and (A.6) with those given above in (4.48)–(4.50) (in the course of which we can put $y_- = y\ , z_- = z$). The result is that now (3.15) is generalised to

$$v_+ = F = v_- - 2\sum_{l=0}^{\infty} p_l\,\frac{(-1)^l}{l!}\,\frac{\partial^l g}{\partial z^l}\ , \tag{4.53}$$

with $g = \log(y^2 + z^2)$. This agrees with (3.15) in the special monopole case when $p_0 = -p\ , p_l = 0$ for $l = 1, 2, 3, \ldots$.

We now consider an Aichelburg–Sexl boost of (4.1)–(4.3) in the $-\bar{z}$ direction. We begin again with the curvature tensor $\bar{R}_{ijkl}$ given in Appendix B and transform to the coordinates $\{x, y, z, t\}$ using the Lorentz boost (4.6) (instead of (2.4)). In this case starting with (4.16) with $A_l = \gamma^{-l-1}p_l$ $(l = 0, 1, 2, \ldots)$, as in (4.12),

$$\psi = \sum_{l=0}^{\infty} p_l\,\frac{\gamma^{-2l-2}}{\hat{R}^{l+1}}\,P_l(\hat{w})\ , \tag{4.54}$$

where

$$\bar{r} = \gamma\,\hat{R}\ , \qquad \hat{R} = \sqrt{\gamma^{-2}(x^2 + y^2) + (z - v\,t)^2}\ , \tag{4.55}$$

and

$$\hat{w} = \frac{\bar{z}}{\bar{r}} = \frac{z - v\,t}{\hat{R}}\ . \tag{4.56}$$

Once again we must calculate the first and second derivatives of $\psi\ , \sigma$ with respect to $\{\bar{x}, \bar{y}, \bar{z}\}$ and express them in terms of $\{x, y, z, t\}$ via (4.6) for substitution into the curvature components R_{ijkl} listed in (B.7). Then the limit $v \to 1$ of these components is taken to arrive at $\tilde{R}_{ijkl}$ in this case. The calculation parallels the one above but with some significant differences. To keep this section within reasonable bounds we have given these calculations in Appendix C. The result is that $\tilde{R}_{ijkl}$ given in (C.20) and (C.21) is the same curvature tensor as (3.7) and (3.8). The presence of the higher multipole

moments than the monopole is lost in this boost. This is not surprising because the scaling of A_l in (4.12) reflects the Lorentz contraction of the source in the axial direction. The shape of the source remains the same but scaled down to the boosted observer [15]. As $v \rightarrow 1$ it appears increasingly as a monopole source and hence we recover the original Aichelburg–Sexl result in this case.

5 Discussion

We have been motivated in our approach by the light–like boost of the Coulomb field described in section 2. The main conclusion we have drawn from this is that the physical significance of the boost is deduced from the behavior of the electric field of the charged particle. This has led us to place the emphasis in the gravitational case on the gravitational field described by the Riemann curvature tensor. Incidentally the difficulties inherent in extending the light–like boost to the Reissner–Nordstrom case have been discussed very clearly in [2]. The metric plays a secondary role for us because once we have recognised that the limit is a space–time which is flat to the future and to the past of a null hyperplane and has a curvature tensor with a delta function singularity on the null hyperplane, then we can reconstruct geometrically this curvature tensor by attaching the two halves of flat space–time, on the null hyperplane, with an appropriate matching. In addition the curvature tensor contains more information than the metric. For example in the case of the boosted multipole field described in section 4, the limit (4.52) of the metric tensor involves only the constant p_0 which is a remnant of the monopole part of the original multipole field. The boosted curvature tensor (4.48)–(4.50) in contrast involves all of the constants p_l , $l \geq 0$ and thus inherits a contribution from all of the original multipole moments.

In the axially symmetric case discussed in detail in section 4, with further details in Appendices B and C, there is, as one would expect, a significant difference in the outcome if one considers a light–like boost transverse to the symmetry axis or parallel to the symmetry axis. The results however are consistent with what one would expect: (a) Such a boost parallel to the symmetry axis yields a result identical to the spherically symmetric monopole case, due to Lorentz contraction of the source and (b) the light–like boost transverse to the symmetry axis yields a plane impulsive gravitational wave which has a singularity along a null geodesic generator of the history of the wave which is more severe than in the monopole case (as can be explicitly seen by comparing (4.48)–(4.50) with (3.7) and (3.8)). The space–time with curvature tensor (4.48)–(4.50) can now be used to study the non–axially sym-

metric high speed collision of two sources of the vacuum Weyl gravitational field (4.1)–(4.3). The axially symmetric high speed collision of two such sources is indistinguishable from the high speed collision of two spherically symmetric sources (two black–holes) discussed in [9].

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A Construction of Plane–Fronted Light–Like Signal

When the two halves of Minkowskian space–time $x > t$ and $x < t$ with line–elements (3.10) and (3.12) are attached on the null hyperplane $x = t$ $(u = 0)$ with the matching (3.14) there is in general a $\delta(x - t)$–term in the curvature tensor of the re–attached space–times and also in the Einstein tensor [10]. The coefficients of the delta function can be calculated in terms of drivatives of the function F in (3.14) using the technique of Barrabès and Israel [12]. The coefficients of the delta function in the Einstein tensor of the re–attached space–times have the form $16\pi\, S_{ij}$ where S_{ij} are the components of the surface stress– energy tensor of the light–like shell. In the case of (3.10) and (3.12) with the matching (3.14) the non–vanishing components of S_{ij} in the coordinates $\{x_-, y_-, z_-, t_-\}$ (with $x_- - t_- = u\ ,\ x_- + t_- = 2v_-$) are given by

$$16\pi\, S_{11} = 16\pi\, S_{44} = -16\pi\, S_{14} = -\frac{2}{F_{v_-}}\left(F_{y_-y_-} + F_{z_-z_-}\right)\ , \tag{A.1}$$

$$16\pi\, S_{12} = -16\pi\, S_{24} = -\frac{2\, F_{y_-v_-}}{F_{v_-}}\ , \tag{A.2}$$

$$16\pi\, S_{13} = -16\pi\, S_{34} = -\frac{2\, F_{z_-v_-}}{F_{v_-}}\ , \tag{A.3}$$

$$16\pi\, S_{22} = 16\pi\, S_{33} = \frac{2\, F_{v_-v_-}}{F_{v_-}}\ . \tag{A.4}$$

The subscripts on F indicate partial derivatives and $F_{v_-} \neq 0$. If the components S_{ij} vanish, as they must for the examples given in sections 3 and

4 above, then the Riemann tensor of the re–attached space–times has the non–identically vanishing components

$$\begin{aligned}
\tilde{R}_{1212} &= \tilde{R}_{2424} = -\tilde{R}_{1313} = -\tilde{R}_{3434} = \\
\tilde{R}_{3134} &= -\tilde{R}_{2124} = \frac{F_{y_-y_-}}{F_{v_-}}\,\delta(x-t)\ , \qquad \text{(A.5)}\\
\tilde{R}_{1213} &= \tilde{R}_{2434} = -\tilde{R}_{3124} = -\tilde{R}_{2134} \\
&= \frac{F_{y_-z_-}}{F_{v_-}}\,\delta(x-t)\ . \qquad \text{(A.6)}
\end{aligned}$$

When applying these formulas, for example in section 3, we can put $y_- = y\ , z_- = z$ and the equations to be satisfied by $F(v_-, y, z)$ are

$$F_{yy} + F_{zz} = 0\ , \qquad F_{yv_-} = 0\ , \qquad F_{zv_-} = 0\ , \qquad F_{v_-v_-} = 0\ , \qquad \text{(A.7)}$$

and

$$\frac{F_{yy}}{F_{v_-}} = -\frac{4p\,(y^2 - z^2)}{(y^2+z^2)^2}\ , \qquad \frac{F_{yz}}{F_{v_-}} = -\frac{8p\,yz}{(y^2+z^2)^2}\ , \qquad \text{(A.8)}$$

The general solution is easily found to be

$$v_+ = F(v_-, y, z) = v_- + 2p\,\log(y^2+z^2)\ , \qquad \text{(A.9)}$$

modulo a (trivial) change of affine parameter $v_+ \to a_1\,v_+ + a_2\,y + a_3\,z + a_4$ where a_1, a_2, a_3, a_4 are real constants. The continuous form of the metric for a general plane–fronted light–like signal is given in [10].

B Curvature Tensor of the Weyl Space–Times

Let $\bar{R}_{ijkl}$ denote the curvature tensor components for the space–time with line–element (4.5) in the coordinates $\{\bar{x}^i\} = \{\bar{x}^A\ , \bar{x}^3 = \bar{z}\ , \bar{x}^4 = \bar{t}\}$ with $A = 1, 2$. The non–identically vanishing components in a convenient form for our purposes are

$$\begin{aligned}
\bar{R}_{ABCD} &= {}^{(2)}\bar{R}_{ABCD} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\left(\frac{\partial g_{AD}}{\partial\bar{z}}\frac{\partial g_{BC}}{\partial\bar{z}} - \frac{\partial g_{AC}}{\partial\bar{z}}\frac{\partial g_{BD}}{\partial\bar{z}}\right)\ , \\
\bar{R}_{ABC3} &= \frac{1}{2}\frac{\partial}{\partial\bar{z}}\left(\frac{\partial g_{BC}}{\partial\bar{x}^A} - \frac{\partial g_{AC}}{\partial\bar{x}^B}\right) + \frac{1}{2}\,\mathrm{e}^{2\psi-2\sigma}\left(\frac{\partial g_{33}}{\partial\bar{x}^B}\frac{\partial g_{AC}}{\partial\bar{z}} - \frac{\partial g_{33}}{\partial\bar{x}^A}\frac{\partial g_{BC}}{\partial\bar{z}}\right) \\
&\quad + \frac{1}{2}\,g^{EF}\left(\frac{\partial g_{AE}}{\partial\bar{z}}\,[BC, F] - \frac{\partial g_{BF}}{\partial\bar{z}}\,[AC, E]\right)\ ,
\end{aligned}$$

$$
\begin{aligned}
\bar{R}_{3434} &= -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial \bar{z}^2} - \frac{1}{4} g^{EF}\,\frac{\partial g_{33}}{\partial \bar{x}^E}\,\frac{\partial g_{44}}{\partial \bar{x}^F} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\frac{\partial g_{33}}{\partial \bar{z}}\,\frac{\partial g_{44}}{\partial \bar{z}} \\
&\quad -\frac{1}{4}\mathrm{e}^{-2\psi}\,\left(\frac{\partial g_{44}}{\partial \bar{z}}\right)^2 \ , \\
\bar{R}_{A3B3} &= -\frac{1}{2}\left(\frac{\partial^2 g_{AB}}{\partial \bar{z}^2} + \frac{\partial^2 g_{33}}{\partial \bar{x}^A \partial \bar{x}^B}\right) + \frac{1}{4} g^{EF}\,\frac{\partial g_{AE}}{\partial \bar{z}}\,\frac{\partial g_{BF}}{\partial \bar{z}} + \frac{1}{2} g^{EF}\,[AB,E]\,\frac{\partial g_{33}}{\partial \bar{x}^F} \\
&\quad +\frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\left(\frac{\partial g_{33}}{\partial \bar{x}^A}\,\frac{\partial g_{33}}{\partial \bar{x}^B} + \frac{\partial g_{AB}}{\partial \bar{z}}\,\frac{\partial g_{33}}{\partial \bar{z}}\right) \ , \\
\bar{R}_{A434} &= -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial \bar{z}\partial \bar{x}^A} + \frac{1}{4} g^{EF}\,\frac{\partial g_{AE}}{\partial \bar{z}}\,\frac{\partial g_{44}}{\partial \bar{x}^F} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\frac{\partial g_{33}}{\partial \bar{x}^A}\,\frac{\partial g_{44}}{\partial \bar{z}} \\
&\quad -\frac{1}{4}\mathrm{e}^{-2\psi}\,\frac{\partial g_{44}}{\partial \bar{x}^A}\,\frac{\partial g_{44}}{\partial \bar{z}} \ , \\
\bar{R}_{A4B4} &= -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial \bar{x}^A \partial \bar{x}^B} + \frac{1}{2} g^{EF}\,[AB,E]\,\frac{\partial g_{44}}{\partial \bar{x}^F} - \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\frac{\partial g_{44}}{\partial \bar{z}}\,\frac{\partial g_{AB}}{\partial \bar{z}} \\
&\quad -\frac{1}{4}\mathrm{e}^{-2\psi}\,\frac{\partial g_{44}}{\partial \bar{x}^A}\,\frac{\partial g_{44}}{\partial \bar{x}^B} \ . \qquad \text{(B.1)}
\end{aligned}
$$

Here $[AB,C]$ is the Christoffel symbol

$$
[AB,C] = \frac{1}{2}\left(\frac{\partial g_{CA}}{\partial \bar{x}^B} + \frac{\partial g_{BC}}{\partial \bar{x}^A} - \frac{\partial g_{AB}}{\partial \bar{x}^C}\right) \ , \qquad \text{(B.2)}
$$

and

$$
\begin{aligned}
{}^{(2)}\bar{R}_{ABCD} &= \frac{1}{2}\left(g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC}\right) \\
&\quad + g^{EF}\left([AD,E]\,[BC,F] - [AC,E]\,[BD,F]\right) \ , \qquad \text{(B.3)}
\end{aligned}
$$

with the comma denoting partial derivative with respect to $\bar{x}^A$.

In the following two paragraphs we list some formulas which are required in section 4 above for the Aichelburg–Sexl boosts of $\bar{R}_{ijkl}$ in the x and z directions (i.e. transverse and parallel to the axis of symmetry respectively).

For a Lorentz boost in the x–direction, $\bar{x}^i \to x^i$ with

$$
\bar{x}^1 = \gamma\,(x^1 - v\,x^4) \ , \qquad \bar{x}^a = x^a \ , \qquad \bar{x}^4 = \gamma\,(x^4 - v\,x^1) \ . \qquad \text{(B.4)}
$$

Here $\gamma = (1-v^2)^{-1/2}$ and Latin letters $a, b, c, \ldots$ take values $2, 3$. Under this transformation the components R_{ijkl} of the Riemann tensor in the unbarred coordinates are related to the barred components (B.1) by

$$
\begin{aligned}
R_{abcd} &= \bar{R}_{abcd} \ , \qquad R_{ab14} = \bar{R}_{ab14} \ , \qquad R_{1414} = \bar{R}_{1414} \ , \\
R_{abc1} &= \gamma\,\left(\bar{R}_{abc1} - v\,\bar{R}_{abc4}\right) \ , \quad R_{abc4} = \gamma\,\left(\bar{R}_{abc4} - v\,\bar{R}_{abc1}\right) \ ,
\end{aligned}
$$

$$
\begin{aligned}
R_{a1b1} &= \gamma^2\left(\bar{R}_{a1b1} - v\,\bar{R}_{a1b4} - v\,\bar{R}_{a4b1} + v^2\bar{R}_{a4b4}\right)\ ,\\
R_{a1b4} &= \gamma^2\left(\bar{R}_{a1b4} - v\,\bar{R}_{a1b1} - v\,\bar{R}_{a4b4} + v^2\bar{R}_{a4b1}\right)\ ,\\
R_{a4b4} &= \gamma^2\left(\bar{R}_{a4b4} - v\,\bar{R}_{a1b4} - v\,\bar{R}_{a4b1} + v^2\bar{R}_{a1b1}\right)\ ,\\
R_{a114} &= \gamma\,\left(\bar{R}_{a114} - v\,\bar{R}_{a414}\right)\ ,\quad R_{a414} = \gamma\,\left(\bar{R}_{a414} + v\,\bar{R}_{a141}\right)\ . \qquad \text{(B.5)}
\end{aligned}
$$

When (B.1) is substituted into (B.5), the barred coordinates are expressed in terms of the unbarred coordinates using (B.4).

For a Lorentz boost in the z–direction $\bar{x}^i \rightarrow x^i$ with

$$
\bar{x}^A = x^A\ ,\qquad \bar{x}^3 = \gamma\,(x^3 - v\,x^4)\ ,\qquad \bar{x}^4 = \gamma\,(x^4 - v\,x^3)\ , \qquad \text{(B.6)}
$$

and $A = 1, 2$. The components R_{ijkl} of the Riemann tensor in the unbarred coordinates are related to the barred components (B.1) by

$$
\begin{aligned}
R_{ABCD} &= \bar{R}_{ABCD}\ ,\qquad R_{AB34} = \bar{R}_{AB34}\ ,\qquad R_{3434} = \bar{R}_{3434}\ ,\\
R_{ABC3} &= \gamma\,\left(\bar{R}_{ABC3} - v\,\bar{R}_{ABC4}\right)\ ,\quad R_{ABC4} = \gamma\,\left(\bar{R}_{ABC4} - v\,\bar{R}_{ABC3}\right)\ ,\\
R_{A3B3} &= \gamma^2\left(\bar{R}_{A3B3} - v\,\bar{R}_{A3B4} - v\,\bar{R}_{A4B3} + v^2\bar{R}_{A4B4}\right)\ ,\\
R_{A3B4} &= \gamma^2\left(\bar{R}_{A3B4} - v\,\bar{R}_{A3B3} - v\,\bar{R}_{A4B4} + v^2\bar{R}_{A4B3}\right)\ ,\\
R_{A4B4} &= \gamma^2\left(\bar{R}_{A4B4} - v\,\bar{R}_{A3B4} - v\,\bar{R}_{A4B3} + v^2\bar{R}_{A3B3}\right)\ ,\\
R_{A334} &= \gamma\,\left(\bar{R}_{A334} - v\,\bar{R}_{A434}\right)\ ,\quad R_{A434} = \gamma\,\left(\bar{R}_{A434} + v\,\bar{R}_{A343}\right)\ . \qquad \text{(B.7)}
\end{aligned}
$$

When (B.1) is substituted into (B.7), the barred coordinates are expressed in terms of the unbarred coordinates using (B.6).

C Boosting in the $-\bar{z}$ Direction

Starting with (4.16) but now with $A_l = \gamma^{-l-1}\,p_l$ for $l = 0, 1, 2, \ldots$, we arrive at ψ given by (4.54) with $\hat{R}, \hat{w}$ as in (4.55) and (4.56). The analogues of (4.19)–(4.21) are

$$
\begin{aligned}
\frac{\partial\psi}{\partial\bar{x}} &= -\sum_{l=0}^{\infty} p_l\,x\,\frac{\gamma^{-2l-4}}{\hat{R}^{l+3}}\,P'_{l+1}\ ,\\
\frac{\partial\psi}{\partial\bar{y}} &= -\sum_{l=0}^{\infty} p_l\,y\,\frac{\gamma^{-2l-4}}{\hat{R}^{l+3}}\,P'_{l+1}\ , \qquad \text{(C.1)}\\
\frac{\partial\psi}{\partial\bar{z}} &= -\sum_{l=0}^{\infty} p_l\,(l+1)\,\frac{\gamma^{-2l-3}}{\hat{R}^{l+2}}\,P_{l+1}\ ,
\end{aligned}
$$

where now the argument of the Legendre polynomials is $\hat{w}$ in (4.56) and the prime denotes differentiation with respect to $\hat{w}$. When the relations (4.22), (4.23) satisfied by the Legendre polynomials are used again here with w replaced by $\hat{w}$. We find that

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial\bar{x}^2} &= -\sum_{l=0}^{\infty} p_l \frac{\partial}{\partial x}\left(\frac{x\,\gamma^{-2l-4}}{\hat{R}^{l+3}}\,P'_{l+1}\right) , \\
\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}} &= -\sum_{l=0}^{\infty} p_l \frac{\partial}{\partial y}\left(\frac{x\,\gamma^{-2l-4}}{\hat{R}^{l+3}}\,P'_{l+1}\right) , \\
\frac{\partial^2\psi}{\partial\bar{y}^2} &= -\sum_{l=0}^{\infty} p_l \frac{\partial}{\partial y}\left(\frac{y\,\gamma^{-2l-4}}{\hat{R}^{l+3}}\,P'_{l+1}\right) , \\
\frac{\partial^2\psi}{\partial\bar{z}^2} &= \sum_{l=0}^{\infty} p_l\,(l+1)\,(l+2)\,\frac{\gamma^{-2l-4}}{\hat{R}^{l+3}}\,P_{l+2} , \\
\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{z}} &= -\sum_{l=0}^{\infty} p_l\,(l+1)\,\frac{\partial}{\partial x}\left(\frac{\gamma^{-2l-3}}{\hat{R}^{l+2}}\,P_{l+1}\right) , \\
\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}} &= -\sum_{l=0}^{\infty} p_l\,(l+1)\,\frac{\partial}{\partial y}\left(\frac{\gamma^{-2l-3}}{\hat{R}^{l+2}}\,P_{l+1}\right) .
\end{aligned}
\tag{C.2}
$$

Limits involving these quantities as $v \to 1$ are required. In section 4 two results, (4.30) and (4.35), played a key role. A substantial difference in the present case is that these are replaced by the following:

$$\lim_{v\to 1} \frac{\gamma^{-2}}{\hat{R}^{l+1}}\,P_l(\hat{w}) = 0 , \qquad l = 0, 1, 2, \ldots , \tag{C.3}$$

$$\lim_{v\to 1} \frac{\gamma^{-4}}{\hat{R}^{l+3}}\,P'_{l+1}(\hat{w}) = 0 , \qquad l = 1, 2, 3, \ldots . \tag{C.4}$$

Both of these are established by induction on l. For (C.3) the result holds for $l = 0$ since

$$\lim_{v\to 1} \frac{\gamma^{-2}}{\hat{R}} = 0 , \tag{C.5}$$

remembering that $\hat{R}$ is given by (4.55). Assume (C.3) holds for l and differentiate it with respect to z to obtain

$$\lim_{v\to 1} \frac{\gamma^{-2}}{\hat{R}^{l+2}}\,\left\{(l+1)\,\hat{w}\,P_l + (\hat{w}^2 - 1)\,P'_l\right\} = 0 . \tag{C.6}$$

Now using (4.23) with w replaced by $\hat{w}$ this becomes

$$\lim_{v\to 1} \frac{\gamma^{-2}}{\hat{R}^{l+2}}\,P_{l+1} = 0 , \tag{C.7}$$

showing that (C.3) holds for l replaced by $l+1$. Thus on account of (C.5) and (C.7) we have established that (C.3) holds. A similar approach leads to (C.4). For $l = 1$, (C.4) is true because

$$\lim_{v\to 1} \frac{\gamma^{-4}}{\hat{R}^4}\, P_2' = 3 \lim_{v\to 1} \frac{\gamma^{-4}}{\hat{R}^5}\, (z - v\,t) = 0\ . \tag{C.8}$$

This last equality follows because we can write (3.5) in the form

$$\lim_{v\to 1} \frac{\gamma^{-4}}{\hat{R}^5} = \frac{4}{3}\, \frac{\delta(z-t)}{(x^2+y^2)^2}\ , \tag{C.9}$$

and $(z-t)\,\delta(z-t) = 0$. Now assume (C.4) holds and differentiate it with respect to z to find that

$$\lim_{v\to 1} \frac{\gamma^{-4}}{\hat{R}^{l+4}}\, \left\{-(l+3)\,\hat{w}\,P_{l+1}' + (1-\hat{w}^2)\,P_{l+1}''\right\} = 0\ . \tag{C.10}$$

Using Legendre's differential equation for $P_{l+1}(\hat{w})$ together with (4.22) (with w replaced by $\hat{w}$) we see that

$$-\,(l+3)\,\hat{w}\,P_{l+1}' + (1-\hat{w}^2)\,P_{l+1}'' = -(l+1)\,P_{l+2}'\ , \tag{C.11}$$

and so (C.10) reads

$$\lim_{v\to 1} \frac{\gamma^{-4}}{\hat{R}^{l+4}}\, P_{l+2}' = 0\ . \tag{C.12}$$

Using (C.8) and (C.12) we have established (C.4) by induction on l. We note that (C.4) also holds for $l = 0$ but it is for $l \geq 1$ that we will now make use of it. The limits we shall need are the following: Using (C.3) we have

$$\lim_{v\to 1} \gamma^2 \frac{\partial^2\psi}{\partial\bar{z}^2} = \sum_{l=0}^{\infty} p_l\,(l+1)\,(l+2)\,\gamma^{-2l}\,\left(\lim_{v\to 1} \frac{\gamma^{-2}}{\hat{R}^{l+3}}\, P_{l+2}\right) = 0\ . \tag{C.13}$$

Similarly

$$\lim_{v\to 1} \gamma\, \frac{\partial^2\psi}{\partial\bar{x}\partial\bar{z}} = 0\ . \tag{C.14}$$

Next

$$\lim_{v\to 1} \gamma^2 \frac{\partial^2\psi}{\partial\bar{x}^2} = -\lim_{v\to 1} \sum_{l=0}^{\infty} p_l\, \frac{\partial}{\partial x}\left(\frac{x\,\gamma^{-2l-2}}{\hat{R}^{l+3}}\, P_{l+1}'\right) = -\lim_{v\to 1} p_0\, \frac{\partial}{\partial x}\left(\frac{x\,\gamma^{-2}}{\hat{R}^3}\right)\ , \tag{C.15}$$

by (C.4) and since now (2.10) is replaced by

$$\lim_{v\to 1} \frac{\gamma^{-2}}{\hat{R}^3} = \frac{2\,\delta(z-t)}{x^2+y^2}\ , \tag{C.16}$$

we can write (C.15) as

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{x}^2}=\frac{2\,p_0\,(x^2-y^2)}{(x^2+y^2)^2}\,\delta(z-t)\ . \tag{C.17}$$

Similarly, using (C.4), we arrive at

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}^2} = -\frac{2\,p_0\,(x^2-y^2)}{(x^2+y^2)^2}\,\delta(z-t)\ , \tag{C.18}$$

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}} = \frac{4\,p_0\,x\,y}{(x^2+y^2)^2}\,\delta(z-t)\ . \tag{C.19}$$

As in the case of the boost in the $-\bar{x}$ direction described in section 4, the function σ given by (4.3) does not survive in the limit $v\to 1$. Now $\bar{R}_{ijkl}$ is given by (B.1), R_{ijkl} is given by (B.7) in the present case and from the latter we calculate $\tilde{R}_{ijkl}$ by taking the limit $v\to 1$. We find that the non–identically vanishing components are

$$\begin{aligned}\tilde{R}_{3232} &= \tilde{R}_{2424}=-\tilde{R}_{2324}=-\tilde{R}_{3131}=-\tilde{R}_{1414}=\tilde{R}_{1314}\\ &= \lim_{v\to 1}2\,\gamma^2\frac{\partial^2\psi}{\partial\bar{y}^2}=-\frac{4\,p_0\,(x^2-y^2)}{(x^2+y^2)^2}\,\delta(z-t)\ ,\end{aligned} \tag{C.20}$$

and

$$\tilde{R}_{3231}=\tilde{R}_{2414}=-\tilde{R}_{1324}=-\tilde{R}_{2314}=\lim_{v\to 1}2\gamma^2\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}}=\frac{8\,p_0\,x\,y}{(x^2+y^2)^2}\,\delta(z-t)\ . \tag{C.21}$$

This is essentially the same curvature tensor as in (3.7) and (3.8) in the Aichelburg–Sexl monopole case. With $p=-p_0$ and with x and z interchanged in (3.7) and (3.8) we obtain (C.20) and (C.21).